Question 1

Q: The joint density of X and Y is $f(x,y) = x + y$ if $0 < x < 1, 0 < y < 1$ and zero otherwise.
(a) Are X and Y independent? (b)Find the density of X and (c) Find $P(X + Y < 1)$.

A: We find densities first and directly see they are not independent. We have $f(x) = \int_0^1 f(x,y)dy = x + \frac{1}{2}, 0 < x < 1$. Similarly $f(y) = y + \frac{1}{2}$ for $0 < y < 1$ and zero else. Since $f(x,y) \neq f(x)f(y)$ in general, we have they are not independent.

To find the probability of $X + Y < 1$, we integrate the joint density of X and Y under the appropriate region (here triangle : $x+y \leq 1$.) Thus $P(X+Y < 1) = \int_0^1 \int_0^{1-y} f(x,y)dxdy = \int_0^1 \frac{(1-y)^2}{2} + y(1-y)dy = \frac{1}{3}$.

Question 2

Q: Let Y and X be independent random variables having respectively exponential distribution with parameter $\lambda > 0$ and uniform distribution over $(0,1)$. Find the distribution of (a) $Z = X + Y$ (b) $Z = X/Y$.

A: (a) Let $f_Z(z)$ denote the density of Z. Using the convolution formula, we have

$$f_Z(z) = \int_0^z f_X(t)f_Y(z-t)dt.$$

We note that $f_X(t) = 1$ if $0 \leq t \leq 1$ and $= 0$ else. So if $z > 1$, we have

$$f_Z(z) = \lambda \int_0^1 e^{-\lambda z}e^{\lambda t}dt = e^{-\lambda z}(e^{\lambda} - 1).$$

If $z \leq 1$, we have

$$f_Z(z) = \lambda \int_0^z e^{-\lambda z}e^{\lambda t}dt = (1 - e^{-\lambda z}).$$

(b) Conditioning on $Y = y$ gives, $\mathbb{P}(Z \leq z) = \int_0^\infty \mathbb{P}(X \leq yz)f_Y(y)dy$. Differentiating with respect to z gives $f_Z(z) = \int_0^\infty zf_X(zy)f_Y(y)dy = \int_0^{1/z} z\lambda e^{-\lambda y}dy = z(1 - e^{-\lambda/z})$.

Question 3

Q: Choose a number X at random from the set of numbers $\{1,2,3,4,5\}$. Now choose a number at random from the subset no larger than X.; i.e., from $\{1,2,...,X\}$. Call this second number Y. (a) Find the joint mass function of X and Y. (b) Find the conditional mass function of X given that $Y = 1$. (c) Are X and Y independent?

A: (a) We have $P(X=j, Y=k) = P(Y=k|X=j)P(X=j) = \frac{1}{5j}$. Since given $X=j$ we know Y is uniform in $\{1,2,..,j\}$. Here we have $k \leq j$. Thus

$$P(Y=1) = \sum_{j=1}^{5} \frac{1}{5j} = c. \tag{1}$$

(b) We use Bayes formula to get $P(X=j|Y=1) = P(Y=1|X=j)\frac{P(X=j)}{P(Y=1)} = \frac{1}{5j}\frac{1}{c}$ where c as in (1).

(c) Not independent: $P(X=1, Y=1) = \frac{1}{5}$. But $P(X=1)P(Y=1) = \frac{c}{5} \neq \frac{1}{5}$.

Question 4

Q: The joint density of X and Y is $f(x,y) = c(x^2-y^2)e^{-x}, 0 \leq x < \infty, -x \leq y \leq x$. Find the conditional distribution of Y, given $X = x$.

A: We use the formula $f(y|x) = \frac{f(x,y)}{f(x)}$. Here $f(x) = \int f(x,y)dy = \int_{-x}^{x} c(x^2-y^2)e^{-x}dx = \frac{2c}{3}x^3e^{-x}, 0 < x < \infty$.

So $f(y|x) = \frac{3}{2x^3}(x^2-y^2), -x \leq y \leq x$. So integrating gives for $t \leq x$, $P(Y \leq t|X=x) = \int_{-x}^{t} f(x,y)dy = \frac{3}{2x^3}(x^2(t+x) - \frac{t^3}{3} + \frac{x^3}{3})$. If $t < -x$, then $P(Y \leq t|X=x) = 0$ and for $t > x$,$P(Y \leq t|X=x) = 1$.

Question 5

Q: Suppose F is a cumulative distribution function (cdf) and $X \sim F(x)$. (a) Verify $G(x) = F^n(x), H(x) = 1-(1-F(x))^n$ are also cdfs, n positive integer. (b) Show there are random variables Y and Z such that $Y \sim G$ and $Z \sim H$.

A: (a) Allowing $x \to \infty$, we have $G(\infty) = 1 = H(\infty)$ and similarly $G(-\infty) = 0 = H(-\infty)$. Also G is increasing since F is increasing. Similarly H is also increasing since $1-F$ is decreasing. Thus G and F are cdfs.

(b) Let $X_1, ..., X_n$ be iid random variables with cdf $F(x)$. Let $Y = \max_{1\leq i\leq n} X_i$ and $Z = \min_{1\leq i\leq n} X_i$. We have $P(Y \leq x) = P(X_1 \leq x, X_2 \leq x, ..., X_n \leq x) = P(X_1 \leq x)^n$. Since if $Y \leq x$ then all X_i are less than or equal to x. The second relation is because $X_1, ..., X_n$ are independent.

Similarly if $Z > x$ then each of $X_i > x$ and we have $P(Z > x) = P(X_1 > x)^n$ again using independence. Thus $P(Z \leq x) = 1-(1-F(x))^n$.

Question 6

Q: Suppose n people are distributed at random along a road L miles long. Let $D \leq \frac{L}{n-1}$. Show that the probability that no two people are less than a distance of D miles apart is, $(1-(n-1)D/L)^n$. What if $D > L/(n-1)$?

A: For any fixed person P, let $x(P)$ denote the position of P. We would like the "$D-$neighbourhood" of each person to be free of any other person (e.g. ball of diameter D centred around each person).

So in particular, the area available for P is $(n-1)D$ out of total area of L. Thus the probability that P falls in the "favourable area" is $1 - \frac{D(n-1)}{L}$. This being true for all n persons each of which is independently distributed, we have the answer. If $D > L/(n-1)$, then $D-$neighbourhood of P will necessarily intersect the $D-$neighbourbood of some other person (since there is not enough space for all the remaining $(n-1)$ $D-$neighbourhoods to be disjoint.) So the probability is zero (note the second part did not need calculation of probability).

Question 7

Q: Let $X_1, ..., X_n$ be iid random variables having distribution function F and density f. The quantity $M = (X_{(1)}+X_{(n)})/2$ defined to be the average of the smallest and the largest value, is called the midrange. Show its distribution is

$$F_M(m) = n\int_{-\infty}^{x} (F(2m-x) - F(x))^{n-1} f(x)dx.$$

A: Let $F_M(m) = P(M \leq m) = P(X_{(n)} + X_{(1)} \leq 2m)$. We condition on the event E_j that X_j is the minimum. So we have

$$P(M \leq m|E_1) = P(X_{(n)} + X_{(1)} \leq 2m|E_1) = \int_{-\infty}^{m} P(X_{(n)} \leq 2m - x|E_1, X_1 = x)f(x)dx.$$

Here we integrate only up to m since minimum always less than or equal to midrange. But if $X_1 = x$ is the minimum, then every other X_i lies between x and $2m - x$. Thus, the last integral equals $\int_{-\infty}^{m}(F(2m-x) - F(x))^{n-1}f(x)dx$. Finally, each X_i takes on the minimum value with equal probability. Therefore $P(M \leq m) = n\int_{-\infty}^{x}(F(2m-x) - F(x))^{n-1}f(x)dx$.

Question 8

Q: If X and Y are independent standard normal random variables, determine the joint density function of $U = X, V = X/Y$. Hence show that X/Y has a Cauchy distribution.

A: We use $f_{UV}(u, v) = \frac{f_{XY}(u,u/v)}{|J|}$, where $J = u_x v_y - v_x u_y$ is the Jacobian of the transformation. Here u_x, v_x etc refer to the partial derivatives. Thus $J = -x/y^2 = -v^2/u$, using the relations $x = u, y = u/v$.

So $f_{UV}(u, v) = \frac{|u|}{2\pi v^2}e^{-\frac{1}{2}u^2(1+v^{-2})}$ and we have $f_V(v) = \int f_{UV}(u, v)du = \int_0^\infty \frac{u}{\pi v^2}e^{-\frac{1}{2}u^2(1+v^{-2})}du$. Using the change of variables $u_1^2 = u^2(1 + v^{-2})$, we have

$$f_V(v) = \int_0^\infty \frac{u_1}{\pi(1+v^2)}e^{-u_1^2/2}du_1 = \frac{1}{\pi(1+v^2)}\int_0^\infty e^{-u_1^2/2}d\left(\frac{u_1^2}{2}\right) = \frac{1}{\pi(1+v^2)}.$$